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NUMERICAL SOLUTIONS OF INTRINSICALLY STEADY-STATE PROBLEMS OF ELECTRODYNAMICS

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ABSTRACT: Questions connected with the numerical solution of intrinsically steady-state problems of electrodynamics are examined in this article. A difference scheme of the second order of accuracy with a positive operator has been derived for the solution of a mixed boundary-value problem. Several iterative operations based on the method of variable directions were used for the solution of a steady-state difference problem. The comparison of different alogrithms was carried out. Solutions for specific problems in the areas of magneto-hydrodynamics, ionization instability in a low-temperature plasma, and magnetic gas dynamics were given as examples.

In this article are discussed problems associated with the numerical /1409* solution of nonself-adjoint boundary-value problems which arise in the investigation of the fields of electric currents or the temperature in a medium with an anisotropic electrical conductivity and thermal conductivity [1]. The use of a variational approach permits transposing the most important properties of the operator of the original problem to the difference operator which approximates it. A divergent difference scheme of the second order of accuracy is constructed for the divergent positive definite operator of the original boundary-value problem. The scheme obtained was used in the numerical solution of a series of specific physical problems.

1. It is required to find a function $u(x_1, x_2)$ which is continuous in the closed rectangular region $\bar{G}$ (see Figure 1) with the boundary $\Gamma = \Gamma_1 + \Gamma_2 + \gamma_1 + \gamma_2$ ($\gamma_1 = AB$, $\gamma_2 = CD$, $\Gamma_1 = AD$, and $\Gamma_2 = BC$) and which satisfies at the interior points of the region G the equation

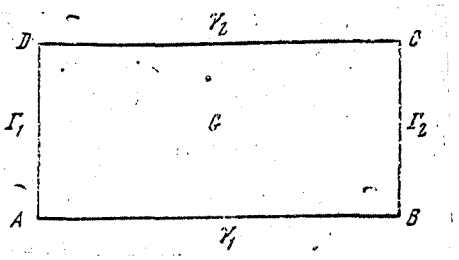

Figure 1.

$$\mathcal{A}u = f, \qquad (1)$$

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$$\mathcal{A}u = -\frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(k\frac{\partial u}{\partial x_2}\right) - \frac{\partial}{\partial x_1}\left(r\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(r\frac{\partial u}{\partial x_1}\right) \tag{2}$$

with the boundary conditions

$$u = 0 \text{ on } \gamma_1 \text{ and } \gamma_2, \tag{3}$$

$$au = k\frac{\partial u}{\partial x_2} - r\frac{\partial u}{\partial x_1} = 0 \text{ on } \Gamma_1 \text{ and } \Gamma_2, \tag{4}$$

$$k(x_1, x_2) \geqslant c_1 > 0, \quad c_1 = \text{const}.$$

The quadratic form

$$[u, u] = \iint_G k\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] dx_1 dx_2 \tag{5}$$

is positive definite

$$[u, u] \geqslant \delta_0(u, u), \quad \delta_0 = \text{const} > 0, \quad (u, v) = \iint_G uv dx_1 dx_2. \tag{6}$$

In the class of sufficiently continuous coefficients k, r and functions u which satisfy (3) and (4), the differential expression (2) defines a linear positive definite operator $\mathcal{A}$ which is self-adjoint only for $r \equiv 0$. A discussion of the problems connected with the existence and uniqueness of the solution of the problem (1) - (4) is contained in the reference [2].

2. Boundary-value problems of the form (1) - (4) are closely associated with the variational principles of describing continuous media [3]; therefore, it is natural to use a variational approach [4] based on determining the solution of the problem with the help of integral identities.

Let $C_1^{(2)}$ be a family of functions which are doubly continuously differentiable within G and which satisfy the boundary conditions (3). The function $u \in C_1^{(2)}$ is the solution of the problem (1) - (4) if and only if for any $v \in C_1^{(2)}$ it satisfies the identity

$$\Phi(u, v) = [u, v] - (f, v) \equiv 0, \tag{7}$$

$$[u, v] = \iint_G I(u, v)\, dx_1 dx_2, \tag{8}$$

$$I(u, v) = k\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) + r\left(\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_2}\right). \tag{9}$$

The validity of this assertion follows from the first Green equation

$$\iint_G v\mathcal{A}u dx_1 dx_2 = [u, v] - \int_C^D v\left(k\frac{\partial u}{\partial x_2} - r\frac{\partial u}{\partial x_1}\right)dx_1 - \int_A^B v\left(k\frac{\partial u}{\partial x_2} - r\frac{\partial u}{\partial x_1}\right)dx_1 \tag{10}$$

and from the arbitrariness in the choice of $v \in C_1^{(2)}$. In its turn (7) can serve as the definition of a generalized solution of the problem (1) - (4).

3. Positive definiteness and divergence of the operator $\mathcal{A}$ is characteristic for the problem under discussion. It is natural to require that the difference analogue A of the operator $\mathcal{A}$ possess the same properties. Construction of the operator of the difference boundary-value problem which approximates the problem (1) - (4) is carried out on the basis of the Eqs. (7) - (9).

In the rectangle $\bar{G} = (0 \leq x_\alpha \leq l_\alpha,\ \alpha = 1, 2)$ we construct the network

$$\bar{\omega}_h = \{x_{1i_1} = i_1 h_1,\ x_{2i_2} = i_2 h;\ i_\alpha = 0, 1, \ldots, N_\alpha,\ h_\alpha = l_\alpha / N_2,\ \alpha = 1,2\}$$

with steps h_1 and h_2 in the directions x_1 and x_2.

We approximate the integral bilinear form [u, v] by the difference expression

$$[u, v]_h = \sum_{i_1=0,\ i_2=0}^{N_1-1,\ N_2-1} (I_h)_{i_1 i_2} h_1 h_2. \tag{11}$$

The approximation of [u, v] can be fulfilled nonuniquely. Among the possible approximations, we select only those which satisfy the following natural requirements: /1411

1) the local approximation of the bilinear form I(u, v) by the difference analogue $I_h(u, v)$ with continuous functions is of the second order:

$$I(u, v) - I_h(u, v) = O(h^2), \qquad h = \max_{\alpha=1,2}(h_1, h_2); \tag{12}$$

2) the quadratic form $[u, v]_h$ is positive:

$$[u, u]_h > 0, \qquad u \neq 0. \tag{13}$$

The bilinear difference functional (11) determines the operator A by means of the identity

$$[u, v]_h = (v, Au)_h = \sum_{i_1=0,\, i_2=0}^{N_1-1,\, N_2} \varkappa_{i_2} v Au h_1 h_2,$$
$$\varkappa_{i_2} = \begin{cases} 1, & i_2 \neq 0, N_2, \\ 0.5, & i_2 = 0, N_2, \end{cases} \tag{14}$$

which is the difference analogue of Green's Eq. (10). The adjoint operator A*

$$[u, v]_h = (u, A^* v)_h = \sum_{i_1=0,\, i_2=0}^{N_1-1,\, N_2} \varkappa_{i_2} u A^* v h_1 h_2.$$

is determined in a similar fashion.

The difference operators A and A* thus constructed are determined by the arbitrary network functions y which satisfy the boundary condition y = 0 at the nodes of the network belong to γ_1 and γ_2 and are divergent. The possibility of constructing difference schemes on the basis of the integral identity (7) was discussed in the references [5, 6].

We used the following approximation to I(u, v) which satisfies (12) and (13):

$$I_h(u, v) = \frac{1}{2} k_{i_1 i_2} \sum_{\alpha=0}^{1} [(u_{x_1} v_{x_1})_{i_1,\, i_2+\alpha} + (u_{x_2} v_{x_2})_{i_1+\alpha, i_2}] +$$
$$+ r_{i_1 i_2} \sum_{\alpha,\, \beta=0}^{1} [(u_{x_2})_{i_1+\alpha,\, i_2} (v_{x_1})_{i_1,\, i_2+\beta} - (u_{x_1})_{i_1,\, i_2+\beta} (v_{x_2})_{i_1+\alpha,\, i_2}].$$

According to (14), we obtain for A the expression

$$Ay = -\sum_{\alpha=1}^{2} (k_- y_{\bar{x}_\alpha})_{x_\alpha} + \frac{1}{2} (p_- y_{\bar{x}_1} + p_+ y_{x_1}) - \frac{1}{2} (q^- y_{\bar{x}_2} + q^+ y_{x_2}) \tag{15}$$

at the interior points, and

$$Ay = -(k_- y_{\bar{x}_1})_{x_1} + \frac{2}{h_2} k^- y_{\bar{x}_2} - q^- y_{\bar{x}_2} - \frac{1}{h_2}(r_- y_{\bar{x}_1} + r_+ y_{x_1}) \tag{16}$$

in the region of the boundary Γ_1 and

$$Ay = -(k_- y_{\bar{x}_1})_{x_1} - \frac{2}{h_2} k^+ y_{x_2} - q^+ y_{x_2} + \frac{1}{h_2}(r_- y_{\bar{x}_1} + r_+ y_{x_1}) \tag{17}$$

in the region of the boundary Γ_2; here $p = \partial r / \partial y$, $q = \partial r / \partial x$. The meaning of the symbols ξ_-, ξ_+, ξ^-, ξ^+ for the functions $\xi(x)$ ($\xi(x)$ is one of the coefficients of k, p, q, and r) is shown in Figure 2. /1412

The difference analog of the identity (7) is

$$\Phi_h(y, w) = [y, w]_h - (f, w)_h \equiv 0, \tag{18}$$

where

$$(f, w) = \frac{1}{4} \sum_{i_1=0,\, i_2=0}^{N_1-1,\, N_2-1} \sum_{\alpha=0}^{1} f_{i_1+\alpha,\, i_2+\alpha} W_{i_1+\alpha,\, i_2+\alpha} h_1 h_2$$

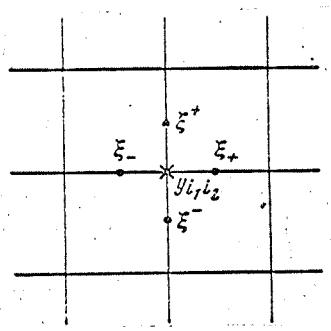

Figure 2.

determines the difference solution y. Finding such a solution is equivalent to a solution of the difference boundary-value problem

$$Ay = f, \tag{19}$$

where the operator A is defined by the Eqs. (15) - (17). Next, we discuss the properties of the operator A.

We note that the inequality of positive definiteness

$$(y, Ay)_h \geqslant \delta \|y\|^2, \quad \delta > 0, \tag{20}$$

where δ = const. and does not depend on h, is valid for the difference operator A just as for the differential operator $\mathcal{A}$.

We introduce into the discussion the self-adjoint operator $A_0 = (A + A^*)/2$ and the skew-symmetric operator $A_1 = (A - A^*)/2$, so that $A = A_0 + A_1$.

The regions of definition of the operators A_0 and A_1 coincide with the region of definition of A. The operator A_0 is positive definite:

$$(y, A_0 y)_h = (y, Ay)_h \geqslant \delta \|y\|^2,$$

which permits us to introduce the norm

$$\|y\|_{A_0} = \sqrt{(y, A_0 y)}.$$

According to (15) - (17),

$$\|y\|^2_{A_0} = h_1 h_2 \left\{ \sum_{i_1=1,\, i_2=1}^{N_1,\, N_2-1} k_-(y_{\bar{x}_2})^2 + \sum_{i_1=1,\, i_2=1}^{N_1-1,\, N_2} k^-(y_{\bar{x}_2})^2 + \frac{1}{2} \sum_{i_1=1}^{N_1} [k_-(y_{\bar{x}_1})^2_{i_2=0} + k_-(y_{\bar{x}_1})^2_{i_2=N_2}] \right\}. \tag{21}$$

To estimate the rate of convergence of the solution y of the difference problem to the solution of the problem (1) - (4), we introduce the error

$$z = y - u. \tag{22}$$

Substituting y = u + z into (19), we obtain for the error z, the equation

$$Az = \psi, \tag{23}$$

where

$$\psi = f - Au = (\mathcal{A}u)_h - Au$$

is the error in the approximation to the differential operator $\mathcal{A}$ by the difference operator A in the solution of u. For a sufficiently continuous solution u of the original problem, the local error in the approximation ψ (in the uniform netric) is a quantity of the order of $O(h^2)$, which follows from the approximation of the bilinear form [u, v] to the order of $O(h^2)$ by the difference analog $[u, v]_h$. /1413

Thus,

$$\|z\|^2_{A_0} = (A_0 z, z)_h = (Az, z)_h = (\psi, z)_h. \tag{24}$$

Using the estimate from the reference [7], we have

$$(\psi, z)_h \leqslant \|\psi\|_{A_0^{-1}} \|z\|_{A_0} \tag{25}$$

and taking (24) into account, we obtain

$$\|z\|_{A_0} \leqslant \|\psi\|_{A_0^{-1}}. \tag{26}$$

From the positive definiteness of $A_0 \geq \delta E$ (E is the identity operator) follows the inequality

$$A_0^{-1} \leqslant \delta^{-1}E,$$

with the use of which we obtain the estimate

$$\|\psi\|^2_{A_0^{-1}} = (A_0^{-1}\psi, \psi) \leqslant \delta^{-1}(\psi, \psi) = \delta^{-1}\|\psi\|^2. \tag{27}$$

From this and from (26) there follows the <u>a priori</u> estimate for the problem (23)

$$\|z\|_{A_0} \leqslant \delta^{-1/2}\|\psi\|. \tag{28}$$

Since $\|\psi\| = O(h^2)$ and $\delta > 0$, the rate of convergence of the difference scheme in the network norm W_2^1 is of the order of $O(h^2)$.

4. Iterative schemes based on the method of variable directions were used in the numerical solution of (19). For this, A was represented in the form of a sum of two uniform operators A_1 and A_2 so that

$$A = A_1 + A_2.$$

A comparison of the two iterative schemes - the longitudinal-transverse scheme (LTS) [8] and the Douglas-Rachford scheme (DRS) [9] - is carried out according to the number of iterations which it is necessary to carry out in achieving the specified accuracy. The calculations were carried out with a constant iterational parameter τ. It is evident from Figure 3, which refers to the solution of (19) for k = r = 1, that for any τ LTS (curve 1) converges in a smaller number of iterations than DRS (curve 2). In Figure 4 are presented the results of the solution of (19) with strongly space-varying coefficients; k = 1, r = a sin γx sin γy + b, and a, b, and γ are constants. It is evident from this that in this case each of the schemes has its own critical value of τ_*, and if this is exceeded, the iterations diverge; for DRS (curve 2), the value of τ_* is greater than for LTS (curve 1). Such a limitation on the parameter τ in the case of strongly varying coefficients appears as a

result of the loss of the fixed sign nature of the operators A_1 and A_2, regardless of the positive definiteness of their sum. For DRS, the curve depicting the dependence of the number of iterations on τ has a "plateau." This situation simplifies the choice of an optimal iterational parameter and permits restriction to a smaller number of iterations in using DRS than for LTS.

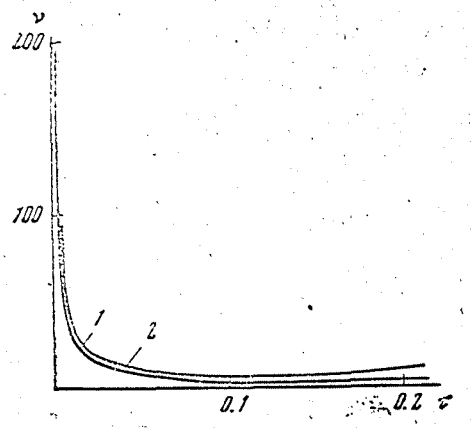

Figure 3.

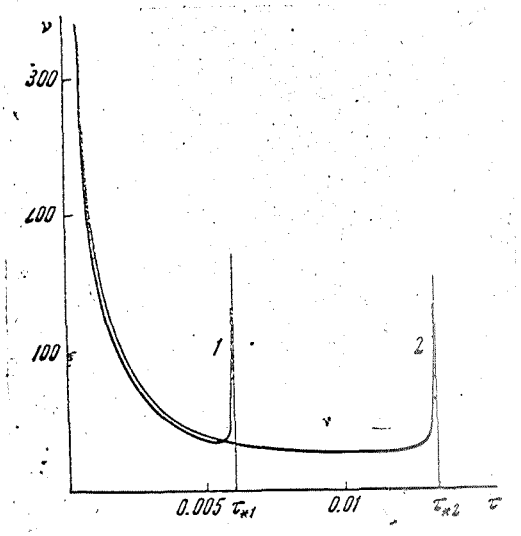

Figure 4.

5. We cite the results of the numerical solution of certain characteristic problems. The description of the results does not pretend to be complete and is of an illustrative nature.

The study of the varied nature of two-dimensional effects in the boundary zones of a magnetic field or near electrodes is of great interest [1]. There is a series of exact solutions in this region for the simplest cases. However, the possibilities of an analytic device are limited even in the case of a very simplified approach to the discussion of problems with constant assigned parameters of the conducting fluid. In the general case of variable coefficients, a numerical solution of the problem is practically the only available approach.

We will discuss the flow of an electrically conducting medium with a constant speed v ($\nu_1 = 1$, $\nu_2 = 0$) and a constant electrical conductivity $\sigma = 1$.

The flow occurs in a thin channel $-\infty \leq x_1 \leq \infty$, $0 \leq x_2 \leq 1$ with parallel sides (see Figure 5), which are everywhere an ideal dielectric, with the exception of the sections occupied by the continuous ideal electrodes, $-0.5 \leq x \leq 0.5$. A specified potential difference $U = 0.7$ is applied between the

electrodes. The magnetic field is constant in the electrode zone and equal to H = 1, but it falls off exponentially with an index 2 outside this zone. The Hall parameter $\Omega = 3H$.

The results of the numerical solution of this problem permit the construction of the distribution of the electric currents which is presented in Figure 4 and also finding such energy integral characteristics as the work capacity of the plasma, the electrical efficiency, and so on. In the reference [10], the results of a series of similar calculations are presented, which have the purpose of clarifying the effect of the spatial distribution of the magnetic field and the electrodes, the Hall parameter, and so forth on the pattern of the current distribution and the integral energy characteristics of the magnetohydrodynamic channels.

The phenomenon of ionization instability in a low-temperature magnetized plasma is of significant interest from the general physical and applied points of view. In the case of specific assumptions, the principal mathematical description of this phenomenon reduces to a solution of the system of equations /1415

$$\frac{\partial}{\partial x_1}\left(\frac{1}{\sigma}\frac{\partial \psi}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{\sigma}\frac{\partial \psi}{\partial x_2}\right)+\frac{\partial}{\partial x_1}\left(\frac{H}{\sigma v}\frac{\partial \psi}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(\frac{H}{\sigma v}\frac{\partial \psi}{\partial x_1}\right)=0, \tag{29}$$

$$\frac{\partial n}{\partial t}=\frac{1}{\sigma}\left[\left(\frac{\partial \psi}{\partial x_1}\right)^2+\left(\frac{\partial \psi}{\partial x_2}\right)^2\right]-nT\nu,$$

where

n is the electron concentration per unit volume, and

$\sigma = n/\nu$ is the electrical conductivity.

The strong dependence of ν on n in the mathematical relationship is a characteristic peculiarity which leads to pronounced spatial variation of the coefficients in the Eq. (29). The numerical solution of the problem has permitted obtaining the qualitative pattern and the quantitative characteristics of the process of the origin and growth of the striae which are observed experimentally. The

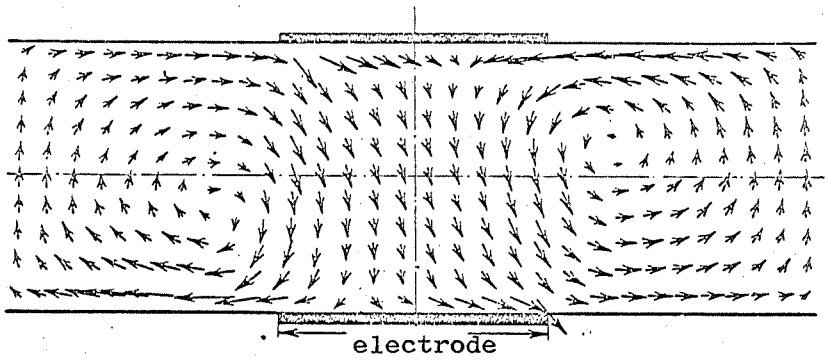

Figure 5.

contour lines for an electron concentration of n = 1 at the time t = 1.8 of the intensive growth of the striae for a Hall parameter value of Ω = 10H are presented in Figure 6. The region of n > 1 (stria) is cross-hatched. The vector field of the density of electric currents is shown by arrows. The tendency towards bunching of the electric current along the strongly ionized striae is distinctly evident. The geometry of the problem is clear from the drawing. The boundary of the region ABCD is an ideal dielectric, with the exception of the completely separated electrodes ab and cd, on which a constant density of the normal component of the electric current $j_{x_2} = -\psi_{x_1} = -1$ is specified. At the initial time t = 0, the electron concentration was constant, n = 1. The connection of the temperature T and the electron concentration n was given by the Saha equation. Collisions of the electrons with ions and the neutral atoms were taken into account. Upon the furthest growth of the process, the distribution of the concentration n and the electric currents ceases to bear a regular nature and resembles the turbulent pattern observed experimentally.

A more detailed analysis of the results of the numerical solution of /141
similar problems is contained in the reference [11].

As a third example, we introduce the result of a numerical solution of the two-dimensional problem of the entrance of a supersonic flow of conducting gas into a magnetic field. This phenomenon is described by the system of equations of magnetic gas dynamics. It is required to find the steady-state solution of the problem which is achieved by integration of the nonsteady state system of equations by the method of adjustment. We pause for the following set-up of the

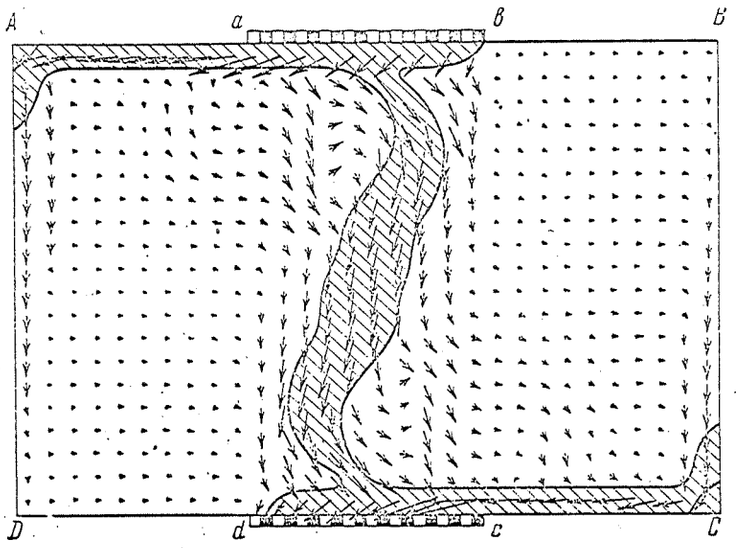

Figure 6.

problem. There is a channel of constant cross section d = 1. The magnetic field is distributed in space in the following manner:

$$H = \begin{cases} 1, & x_1 > 0, \\ e^{1.5x_1}, & x_1 < 0. \end{cases}$$

We limit the region of the solution of the problem to the cross sections $x_1 = \pm 1.5$ (see Figure 7). At the entrance to the channel (cross section $x_1 = -1.5$) the supersonic flux of the gas is specified by the following parameters: the density $\rho = \rho_0$, the temperature $T = T_0$, the velocity $u = u_0$, the pressure $p = p_0$, and the adiabatic constant $\gamma = 1.12$. The conductivity is determined from the electron concentration of a cesium admixture computed according to the Saha equation:

$$\sigma = \sigma_0 \left(\frac{T}{T_0}\right)^{\gamma/4} \left(\frac{\rho}{\rho_0}\right)^{-1/2} \exp\left[-\frac{I}{2}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right].$$

The basic dimensionless parameters which characterize the flow are the Mach number M and the parameter of hydromagnetic interaction R_M. At the cross

section $x_1 = -1.5$, they have the value $M = 2.92$ and $R_M = 0.5$. The steady-state values of the spatially-distributed parameters are presented in Figure 7. The /1417 ratio of the value of the parameter on the contour line to the value of the corresponding parameter at the entrance cross section is indicated. For a more detailed description, see the reference [12]. In Figure 7 is shown the current vortex which arises at the entrance of the conducting medium into the magnetic field. The existence of such a vortex is well known [1] from the solution of this problem in the case of a fixed motion of the medium, i.e., in the approximation $R_M \ll 1$.

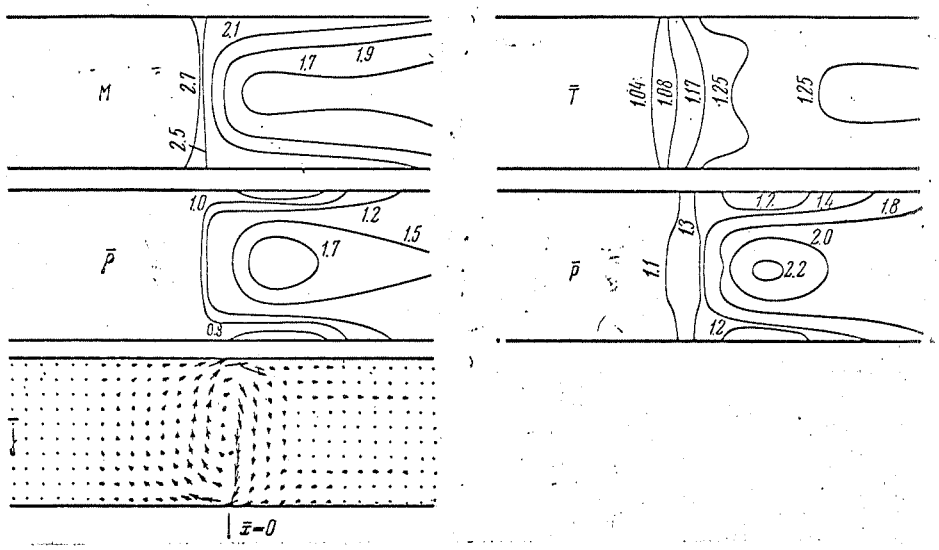

Figure 7.

The course of the contour lines in Figure 7 shows that the injection of the supersonic flux of conducting gas into the magnetic field with $R_M = 0.5$ is accompanied by a significant deceleration of it and a compression of it towards the axis. In its turn the variation of the parameters of the flux leads to a sharp attenuation of the electric current in the zone of the constant magnetic field at $x_1 > 0$.

In conclusion, the authors express their gratitude to I. V. Fryazinov for fruitful discussions and also to T. A. Gorbushina, E. E. Myshetskaya, and V. A. Ravinskaya for developing the programs and carrying out the calculations.

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